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Uncertainty of Volumetric Fraction Estimates Using 2-D Measurements

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ABSTRACT: This paper presents an analytical solution to quantify the uncertainty of volumetric fraction (V_f) estimates in a heterogeneous material using 2-D probes. The analytical solution was derived based upon the concept of representative volume element (RVE). The results show that the uncertainties of the estimates depend upon the size of blocks, measurement area, and level of V_f. The analytical solution has been verified via numerical simulation. The latter was carried out by first generating spherical blocks to various level of V_f, which was then sampled by 2D probes in obtaining the V_f estimates and the associated uncertainties. Application examples are given at the end.

1. INTRODUCTION

The mechanical behavior of a heterogeneous geomaterial depends strongly on the volumetric fraction (V_f) of its various constituents [1-9]. To estimate V_f, one may take sample at random points, along a line, on a planar area, or in a three dimensional region. According to the basic principles of stereology, if features are distributed under isotropic, uniform and random conditions, the results obtained will be the same regardless of the dimension of the probes [10]. In other words, volumetric fraction estimates satisfy the following relationship,

$$P_f = L_f = A_f = V_f \tag{1}$$

where, P_f is point-count fraction, L_f is linear fraction, A_f is area fraction, and V_f is volumetric fraction.

In geotechnical engineering, three geometric probes are often used in estimating the volumetric fractions of heterogeneous geomaterials (i.e. bimrocks), namely, the one dimensional scanlines or boreholes, the two dimensional cross sectional images or window mapping, and the three dimensional sieve analyses. Although sieve analysis is the most accurate method for laboratory-scale studies, it is hindered by the fact that separation of blocks from the weaker matrix is not always possible. Hence, scanline, borehole, window mapping, or image analyses are widely used probes to predict the various V_f measurements of geomaterial. But these probes face a common underlying challenging question: How large should a sample size be?

1-D probe is perhaps the most efficient and economical method for estimating volumetric fractions, and research on what constitute an adequate scanline length, or sample size, can be traced way back to 1961. That was when Hilliard & Cahn [11] made an assumption that intercept length of a scan line on the inclusion followed a Poisson distribution, and obtained an equation to calculate the standard deviation of V_f with respect to sample size. This allows one to select an acceptable error and from that determine the corresponding scanline length. But the Poisson distribution assumption limits Hilliard & Cahn's equation to be applicable only for problems with V_f roughly less than 5% [11]. Medley took a different route by constructing laboratory samples with different volumetric fractions to investigate the effects of sample size and V_f on the uncertainty of V_f estimates [12]. Tien et al. [13] introduced the concept of representative volume element (RVE) and derive an analytical solution as follows,

$$COV_{1D}\sqrt{\frac{L}{D}} = \sqrt{\frac{8}{3\pi \cdot V_f} - \sqrt{\frac{\pi}{4 \cdot V_f}}} \tag{2}$$

where COV_{1D} is coefficient of variance of V_f estimation by using 1-D probe, D is an equivalent diameter of blocks, L is 1-D probe length. To verify the validity of their RVE approach, Tien et al. [14] ran extensive numerical simulation and obtained

$$COV_{1D}\sqrt{\frac{L}{D}} = -0.925(\log V_f)^3 + 4.79(\log V_f)^2 - 10.1(\log V_f) + 8.69 \quad (3)$$

When plotted, the simulation results matched well with those from the analytical solution and thus provided the verification. Tien et al. [15,16] further extended the analytical solution to include anisotropic consideration by adding an anisotropic factor, *K*, and derived the following equation,

$$COV_{1D}\sqrt{\frac{L}{D}} = \frac{1}{K}\sqrt{\frac{8}{3\pi \cdot V_f} - \sqrt{\frac{\pi}{4 \cdot V_f}}} \quad (4)$$

$$K = \sqrt[4]{k\cos^2\theta + \frac{1}{k}\sin^2\theta} \quad (5)$$

where k is an aspect ratio, θ is orientation of blocks.

With the advance in image analysis, 2-D computerized image analysis has emerged as the dominant method in the geotechnical field for estimating V_f [4,5]. In their classical work, Hilliard & Cahn [11] also proposed an analytical solution to address the issue of sample size versus error as follows,

$$COV_{2D} = \frac{(COV_a)^2 + 1}{M_a} \quad (6)$$

where COV_{2D} is coefficient of variation of V_f estimates from a 2-D probe, COV_a is coefficient of variation of intercepted block areas with the probe, and M_a is number of intercepted blocks. Extending that of Eq. (3), Siao [17] investigated the uncertainty of V_f estimates using 2-D probe through simulation,

$$COV_{2D}\sqrt{\frac{A}{\pi R^2}} = 2.35(\log V_f)^2 + 0.151(\log V_f) + 0.223 \quad (7)$$

where R is the equivalent radius of elliptical or circular blocks.

2. ANALYTICAL SOLUTION

The similarity in form between equations (3) and (7) suggests that a similar analytical solution to equation (2) might also be derived for 2D probes. Indeed, this turned out to be the case.

2.1. RVE model

The concept of RVE is illustrated in Fig. 1. This figure shows a 2D probe samples a 3D geomaterial populated with randomly distributed spherical blocks. Our abstraction started by visualizing each block to be enclosed inside a represented volume. This lead us to construct an sphere-cube RVE as illustrated in Fig. 2.

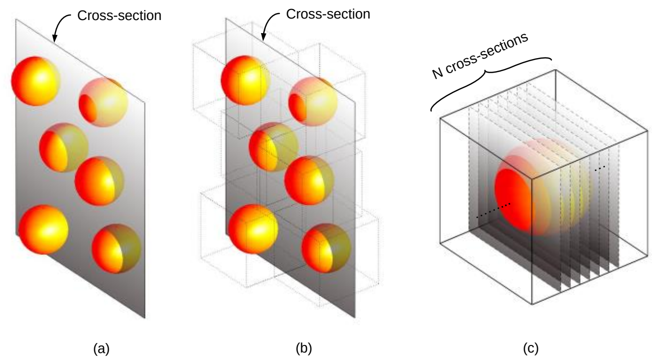

Fig. 1. The concept of RVE. (a) A 2D probe randomly samples a material; (b) each of blocks can be viewed as enclosed in a randomly arranged cube of the same size; (c) the intercepted A_{fs} can be viewed as a random cross-section sampled within a single RVE.

For a given RVE, the volume is selected so that ratio of the block to the volume equals to the expected A_f. The probability distribution of the individual intercepted block area can be obtained by sampling within a given represented volume. The maximum V_f achievable through RVE is π/6.

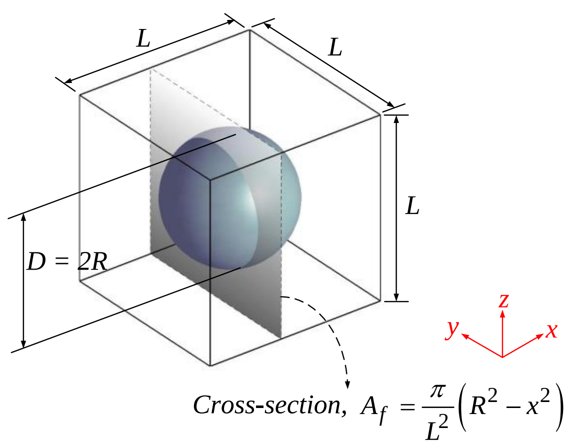

Fig. 2. Sampling within a RVE.

Without losing generality, let a sampling in a RVE be done by a random vertical slicing, and the area ratio for each slicing, A_{fs}, is thus a random variable. If we further let the RVE be a cube with a side length of L, A_{fs} for a random slicing at x can easily be found as

$$A_{fs} = \frac{\pi}{L^2}(R^2 - x^2) \quad (8)$$

when x lies from –R to R; A_{fs} is zero otherwise. The expected value of A_{fs} is,

$$E[A_{fs}] = \frac{4\pi R^3}{3L^3} = V_f \quad (9)$$

The standard deviation of A_{fs} in a RVE can be found as,

$$\sigma_{A_{fs}} = \sqrt{\frac{1}{L}\int_{-L/2}^{L/2}\frac{\pi^2(R^2-x^2)^2}{L^4}dx - V_f^{\,2}} \\ = \sqrt{\frac{16\pi^2R^5}{15L^5} - V_f^{\,2}} \qquad (10)$$

In terms of V_f, $\sigma_{A_{fs}}$ becomes

$$\sigma_{A_{fs}} = \sqrt{\frac{16}{15\pi^{1/3}}\left(\frac{3\cdot V_f}{4}\right)^{5/3} - V_f^{\,2}} \qquad (11)$$

A 2D probe can be viewed as a sum of slice samples, thus its statistics can be found through a sum of the random variable, V_{fs}. In a practical application, V_f would follow the Gaussian distribution as dictated by the Central Limit Theorem. The relationship between the standard deviation of a population and that of a sample of size n is

$$\sigma_{sample} = \frac{\sigma_{population}}{\sqrt{n}} \qquad (12)$$

Here, a population comprises of all the slice samples from a probe, $\sigma_{population}$ is thus $\sigma_{A_{fs}}$; the sample size is the number of RVE can be fitted in the probed area A. That is,

$$n = \frac{A}{L^2} = \frac{A}{\pi R^2}\sqrt[3]{\frac{9\cdot V_f}{16\pi}} \qquad (13)$$

Hence, taking into account that the estimate is unbiased, for a 2D probe, σ_{2D} can be rewritten by substituting Eq. (13) and Eq. (11) into Eq. (12) as follows

$$\sigma_{2D}\sqrt{\frac{A}{\pi R^2}} = \sqrt{\frac{4\cdot V_f}{5} - \sqrt[3]{\frac{16\cdot V_f^{\,4}}{9\pi}}} \qquad (14)$$

2.2. Polydisperse Inclusion

The above derivation assumes that the inclusion consists of only one single size particles. An extension to polydisperse inclusion is detailed herein.

We consider here the inclusion consists of k different particle sizes, each size has a different number of particles present. From the total sum of squares [18], and assume no correlation between particle of different sizes, it then follows,

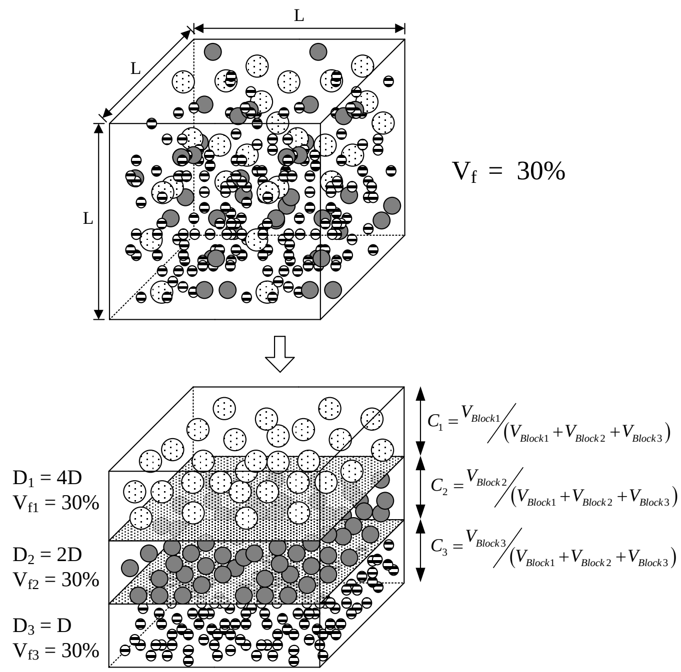

Fig. 3. Model the polydisperse inclusion problem via the concept of RVE.

$$\sigma_{2DP}^{\;2} = \sum_{i=1}^{k} C_i \sigma_{2Di}^{\;2} \qquad (15)$$

where, C_i is the volume fraction of feature i. Defining the mean radius as follows,

$$\bar{R} = \sqrt{\sum_{i=1}^{k} C_i R_i^{\,2}} \qquad (16)$$

and after some manipulation, it follows,

$$\sigma_{2DP}\sqrt{\frac{A}{\pi\bar{R}^2}} = \sqrt{\frac{4\cdot V_f}{5} - \sqrt[3]{\frac{16\cdot V_f^{\,4}}{9\pi}}} \qquad (17)$$

In terms of coefficient of variation, it becomes

$$COV_{2DP}\sqrt{\frac{A}{\pi\bar{R}^2}} = \sqrt{\frac{4}{5\cdot V_f} - \sqrt[3]{\frac{16}{9\pi\cdot V_f}}} \qquad (18)$$

Equations (14), (17) and (18) are the new results obtained by this study.

3. NUMERICAL SIMULATION

Numerical simulation was carried out using spherical particles as inclusions. Periodic boundary was adopted.

A typical simulation involves the following steps:

1. Define a cubic problem domain of B×B×B, and the particle diameter or diameters.

2. Define V_f of the problem.
3. Place particles randomly without overlap until the target V_f is reached.
4. Take 2D probes repeatedly at random location and compute the areal fraction for each probe.
5. A desired areal probe can be obtained as a sum of many probes. A statistical description can be obtained through sampling.

Fig. 4 illustrates the sampling procedure on a material with inclusions of different size particles.

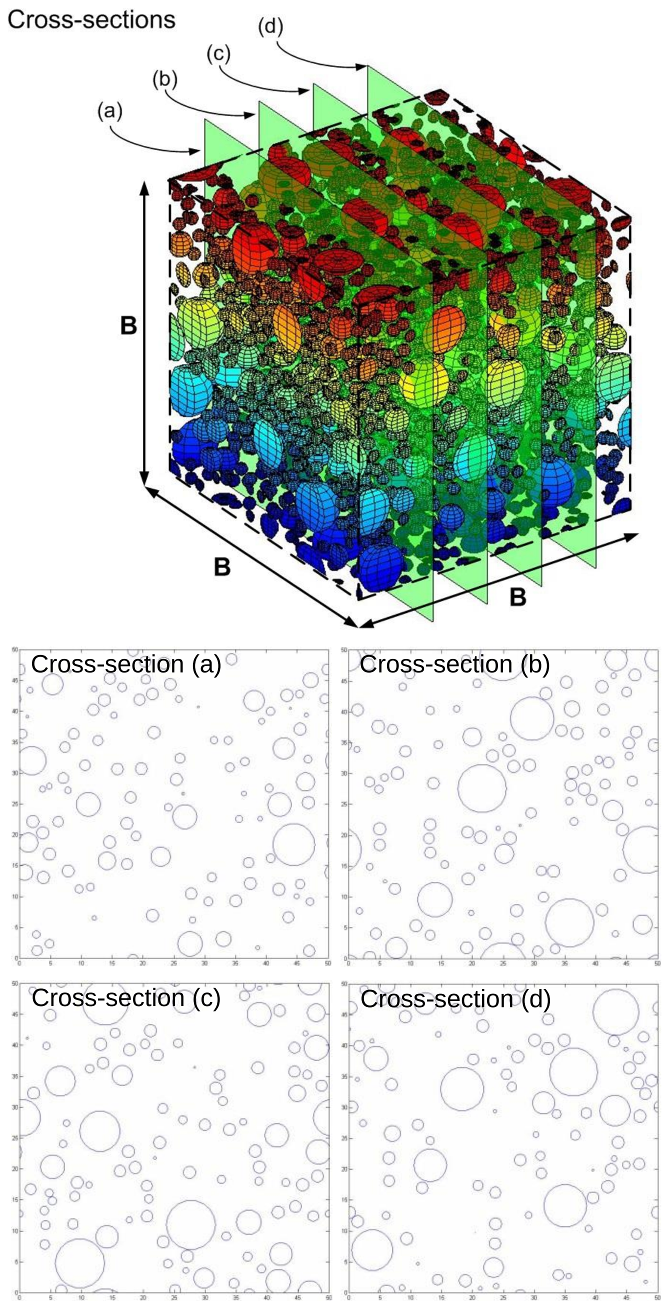

Fig. 4. Random sampling on a material with inclusion of different size particles

The computer simulation study employed both monodisperse particles and polydisperse particles. For the latter, we mixed particles of three different diameters, they are 80 mm, 40 mm and 20mm, respectively. Three fractional ratios were adopted. C_1: C_2: C_3 used were 32: 20: 25, 100: 125: 156 and 32: 20: 5. We also obtained an empirical equation from regression on the simulation results as follows,

$$COV_{2DN}\sqrt{\frac{A}{\pi\bar{R}^2}} = 1.57(\log V_f)^2 - 0.56(\log V_f) + 0.0742 \tag{19}$$

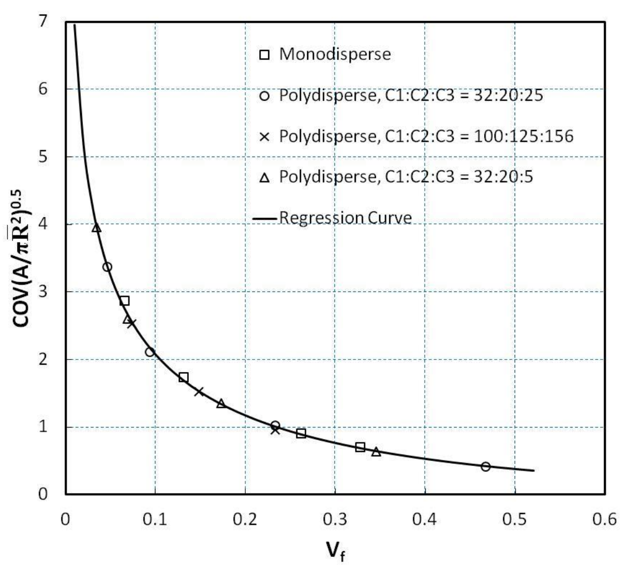

Fig. 5. Result of numerical simulation.

4. RESULT INTEPRETATION

As illustrated in Fig. 6, our simulation results compare well with estimates from the new analytical solution. It can be observed that when V_f is less than 15%, the analytical solution gives slightly smaller error than that from simulation. Whereas when V_f is greater than 15%, the trend reversed. But the differences are rather small for all practical purposes.

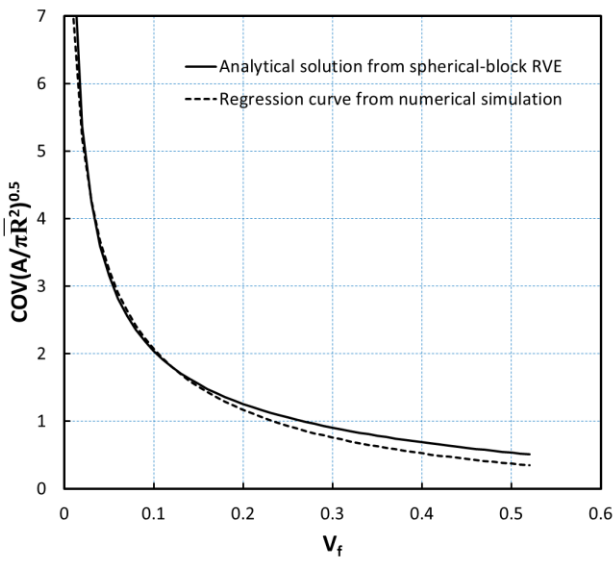

Fig. 6 Comparison between the analytical solution and numerical simulation

5. ILLUSTRATED EXAMPLES

Two examples are presented here to illustrate the application of the analytical solution.

Example 1

Assume that a 2D probe gave an estimate of V_f to be 0.2, and the total probe area, A, was 7854cm^2. Given the mean inclusion particle size $\overline{R}$ to be 10cm, what would be the 95% confidence interval of the V_f estimate？

From the analytical solution of equation (17), one finds the standard error to be

$$\sigma_{2DP} = \sqrt{\frac{10^2\pi}{7854}\cdot\left(\frac{4\cdot 0.2}{5} - \sqrt[3]{\frac{16\cdot 0.2^4}{9\pi}}\right)}$$
$$= 0.0503$$

The 95% confidence interval of V_f estimate then can be derived :

$$V_f - 1.96\cdot\sigma_{2DP} \sim V_f + 1.96\cdot\sigma_{2DP}$$
$$\Rightarrow 0.2 - 1.96\cdot 0.0503 \sim 0.2 + 1.96\cdot 0.0503$$
$$\Rightarrow 0.101 \sim 0.299$$

If, on the other hand, the regression equation from simulation, i.e., equation (19), is used, the standard error would be 0.049, and the interval would lie in the range of 0.103~0.297.

Example 2

If the confidence interval as obtained above is deemed not satisfactory, how large the probe area should be so that the 95% confidence interval would be reduced to 0.150~0.250?

First, we can determine the required standard error as follows,

$$0.2 - 1.96\cdot\sigma_{2DP} = 0.150$$
$$\Rightarrow \sigma_{2DP} = 0.0255$$

Plug in the values of σ_{2DP}, V_f and $\overline{R}$ into equation (17), the required probe area, A , can be readily found as

$$\sigma_{2DP} = \sqrt{\frac{\pi\overline{R}^2}{A}\cdot\left(\frac{4\cdot V_f}{5} - \sqrt[3]{\frac{16\cdot V_f^{\,4}}{9\pi}}\right)}$$
$$0.0255 = \sqrt{\frac{10\pi}{A}\cdot\left(\frac{4\cdot 0.2}{5} - \sqrt[3]{\frac{16\cdot 0.2^4}{9\pi}}\right)}$$
$$\Rightarrow A = 30666\,\mathrm{cm}^2$$

6. CONCLUSIONS

Using both monodisperse particles and polydisperse particles, we have derived a new analytical solution that can be used in quantifying the errors volume fraction estimates by 2D probes. Our solution indicates that error is a function of the fraction ratio and is inversely proportional to the square root of the probe area. The solution obtained has been verified through computer simulation.

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